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SOME SPECTRAL CHARACTERISTICS OF CEPHEID VARIABLES

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In a recent investigation of the absolute magnitudes of 500 stars[1] we have shown that in a large number of cases the intensity of the hydrogen lines is abnormally great in relation to the spectral type as derived from the more general characteristics of the spectrum. The effect is most striking in the case of giant M-type stars such as α Orionis, but it is marked for many K and G-type spectra as well. The suggestion was made that abnormal intensity of the hydrogen lines is a general characteristic of the giant stars of at least some of the spectral types. A discussion of this question with reference to certain of the variable stars of the δ Cephei type is the object of this communication.

It has been shown by the investigations of Hertzsprung[2] and others that the Cepheid variables are stars of very high intrinsic luminosity, with an average absolute magnitude, as derived from parallactic motion, more than seven magnitudes brighter than the sun. Directly measured parallaxes of five stars yield a value of about five magnitudes. Whichever result is accepted it is evident that these stars are exceedingly luminous and form most interesting material for a study of the question of the intensity of the hydrogen lines.

The spectrum of the Cepheid variables has been studied by many observers, and it is quite impossible to make adequate reference to their results in this place. In general, however, most of the work of Albrecht[3], Duncan[4], Shapley[5], and many others has dealt with variations in the character of the spectrum at maximum and minimum of light. For the purpose we have in mind a direct comparison is instituted between the spectra of the several variables and that of the sun, which is selected as a typical star of type G_0. Photographs of the spectra, all of which were made with a slit spectrograph, taken at maximum and minimum of light are also compared directly with one another. Furthermore typical stars having spectra of types F_0, F_5, and G_5 have been

selected from among those classified by the Harvard observers, and these spectra have been used for comparison purposes as well as that of the sun. The spectra have been compared by the aid of a Hartmann spectro-comparator.

A summary of the results is given in brief form in the accompanying table. It became clear from the very first comparisons that the hydrogen lines are abnormal in the Cepheid spectra, being in some cases several times as strong as in the sun, although in other respects the spectra do not differ greatly. We have accordingly adopted the plan used in previous investigations of making two determinations of spectral type. The first is based upon the hydrogen lines alone; the second upon the more general features of the spectra, in particular, the intensities of the arc lines of the various elements, and other characteristics which seem to be primarily a function of general spectral type.

	MAXIMUM OF LIGHT		MINIMUM	
	Hydrogen	General spectrum	Hydrogen	General spectrum
TU Cassiopeiae	F_0	F_9	F_8	F_9
SU Cassiopeiae	F_3	F_7	F_6	F_8
SZ Tauri	F_0	F_8	F_8	F_8
T Monocerotis	F_0	G_0	F_7	G_1
RT Aurigae	F_0	F_7	F_8	G_0
W Geminorum	A_9	G_0	F_6	G_0
ζ Geminorum	F_2	F_9	F_8	G_0
η Aquilae	F_3 (near max)	F_9	F_8	G_1
δ Cephei	F_2 (near max)	F_9	F_9	G_0
Mean	F_1	F_9	F_7	G_0

Three conclusions may be drawn from these results: First, that the hydrogen lines are abnormally strong in all of these stars, the difference translated into spectral type amounting to eight divisions of the Harvard scale at maximum of light: Second, that at minimum this difference is reduced, amounting in the average to but three divisions: Third, that there is little or no difference in the general spectrum at maximum and minimum so far as the criteria here used are concerned. Certain differences in other respects will be referred to presently. The general conclusion may therefore be drawn that the variation in spectral type among the Cepheid variables is mainly a variation in the intensity of the hydrogen lines.

The determination of spectral type of several Cepheid variables by Shapley[5] and of δ Cephei by Adams and Shapley[6], were based wholly on the hydrogen lines, the anomalous behavior of these lines in many stars not being fully recognized when the earlier work was done. A comparison of the present spectral determinations from the hydrogen lines with those of Shapley for seven stars common to the two lists shows a close degree of accordance in the amount of variation at maximum and minimum of light. The values are seven

and eight spectral divisions in the two cases respectively. There is, however, a small systematic difference in spectral type at both maximum and minimum, the types given here being on the average slightly earlier.

It is clear from the very special character of the variations in the spectra of the Cepheid variables that it is doubtful if they can be regarded as furnishing direct evidence as to the order of evolution of stars in general. In the normal succession of spectral types changes in intensity of the hydrogen lines are accompanied by numerous important changes in many features of the spectrum which appear to remain essentially unaltered in the spectra of the Cepheids.

Reference has been made to certain differences in the spectra of these stars at maximum and minimum of light other than those in the hydrogen lines. The most important of these are: (1) a shift of the maximum of the continuous spectrum toward shorter wave-lengths at maximum of light, a result found by Albrecht[5]: (2) a general slight widening of the spectral lines at minimum: (3) an increase in the intensity of the so-called 'enhanced' lines at maximum. The last two characteristics were noted in the investigation of the spectrum of δ Cephei by Adams and Shapley[6]. The change in the enhanced lines is probably most significant in its bearing on the variation of absolute magnitude. The three lines principally used for the determination of absolute magnitudes of stars of this type of spectrum are all strongly enhanced and it seems probable, as we have suggested previously, that enhanced lines as a class vary with luminosity. Accordingly we have compared the intensities of some of the more prominent enhanced lines in the spectra of these variables with their intensities in the solar spectrum, and also on the photographs taken at maximum and minimum of light. In all cases these lines are much more intense in the stellar spectra, and to a less degree more intense at maximum than at minimum of light. Among the lines are the following:

4077 Sr	4233 Fe	4290 Ti	4385 Fe	4584 Fe
4215 Sr	4246 Y	4376 —	4534 Ti	

Of these λ 4077 shows the largest difference, being fully five times as strong in some of the stellar spectra as in the sun. With the aid of the reduction curves for absolute magnitude, the difference in the intensities of the three lines λ 4077, 4215 and 4290 at maximum and minimum of light may be converted into differences of magnitude. They give for the average of the nine stars the values 0.8, 1.3 and 0.6, respectively, or a mean of 0.9. This is in very fair agreement with the average variation in apparent magnitude of 0.7 for the same stars.

Two other features of the spectra of this class of variables may be referred to briefly. The first is the marked difference in the relative intensities of the hydrogen lines when compared with the solar spectrum. In the sun the line $H\beta$ is considerably more intense than $H\gamma$, but in all of the Cepheid spectra the reverse is the case, $H\gamma$ being much stronger than $H\beta$. At minimum of

light this effect is slightly less marked than at maximum. It seems probable that we are dealing here with a shift in the maximum of intensity of the line spectrum of hydrogen similar to that which occurs in the continuous spectrum. It has been shown in the laboratory that in the case of lines belonging to the same series increase of temperature increases the intensity of the more refrangible lines, the maximum moving toward shorter wave-lengths. In such high luminosity stars as the Cepheid variables such a difference as compared with the sun is highly probable.

It is of interest to note in connection with a study of the spectra of these variable stars that the spectra of certain stars which, with the exception of ρ Cassiopeiae, are not known to show light variation have very similar characteristics both as regards the intensity of the hydrogen and the enhanced lines. The most promient cases of this kind are the following:

	R. A.	DEC.	GAL. LONG.	GAL. LAT.
δ Canis Majoris	$7^h\ 4^m$	$-26°\ 14'$	204°	−8°
ξ Puppis	7 45	−24 37	211	0
γ Cygni	20 19	+39 56	45	+2
Boss 5931	22 56	+56 25	75	−5
ρ Cassiopeiae	23 49	+56 57	83	−5

The star δ Canis Majoris has been found at the Lick Observatory to show a small variation in radial velocity. Some of these stars have spectra nearly identical with that of the Cepheid variables, and it is an interesting fact that all of them are situated very near the galactic plane and thus share in the peculiar distribution of the latter.

[1] Adams, W. S., and Joy, A. H., *Mt. Wilson Contr.* No. 142; *Astroph. J., Chicago*, **46**, 1917, (313–339).

[2] Hertzsprung, E., *Astr. Nachr., Kiel*, **196**, 1913, (201–208).

[3] Albrecht, S., *Lick Obs. Bull., Berkeley*, No. 118.

[4] Duncan, J. C., *Ibid.*, No. 151.

[5] Shapley, H., these PROCEEDINGS, **2**, 1916, (208–209); also *Mt. Wilson Contr.* No. 124.

[6] Adams, W. S., and Shapley, H., these PROCEEDINGS, **2**, 1916, (135–142).

TYPES OF ACHROMATIC FRINGES

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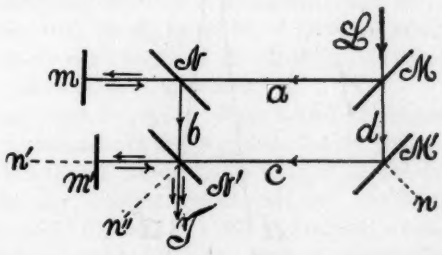
The difficulty of obtaining fringes of the strictly achromatic type (i.e., two strong fringes with a black line between and the remaining fringes green-

reddish and faint) in the rectangular or other interferometer, has been referred to in my earlier papers. As a rule the fringes found are more or less diffuse, non-symmetric, with large numbers (10 or more) about equally strong. Such fringes are, of course, useless in displacement interferometry. When the sharp fringes needed are obtained, their definition is independent of the particular part of any of the glass plates used, and any plate may be rotated 180° in its own plane without spoiling the sharpness of the fringes. Hence such slight curvatures or wedge shapes as the plates may possess, are without influence on the phenomenon. To further test this I devised a screw press adapted to push the vertical edges of a plate to the rear and the middle forward, so as to give the plate marked cylindricity. Quarter and eighth inch plates were operated on, in the latter case sufficiently to give the two superposed slit images quite unequal width; but no essential or useful improvement of the fringes was observed. The type of the fringe was not altered. Again the symmetrical fringes may be obtained from plates thickly or thinly silvered, without essential difference.



It follows therefore, that the relative thickness of the glass paths traversed by the interfering beams can alone be of influence in shaping the fringe pattern in the manner in question. This is in consonance with the general theory of achromatic fringes, the result being a superposition of the color phenomena due to the dispersive refraction of the glass and the colors resulting from the wave lengths of the interfering rays. To test this the annexed form of apparatus (see figure) is particularly convenient, as the fringes are easily found. Moreover both rays, *a* and *c*, from the collimator at *L*, eventually pass through the plate, *N'*, before reaching the telescope at *T*. It is thus merely the thickness of the half silvers, *M* and *N*, both at 45°, that is here in question. If this thickness is the same, the sharp symmetrical design of but two strong fringes appears. If the difference of thickness is but little over half a millimeter, many fringes, non-symmetric in distribution are the rule. If the differential thickness is several millimeters there may be hundreds of fringes. If these are small they may be enlarged at pleasure; but they are always faint and useless for measurement.

[1] Advance note from a Report to the Carnegie Institution of Washington.

INTERFERENCE OF PENCILS WHICH CONSTITUTE THE REMOTE DIVERGENCES FROM A SLIT

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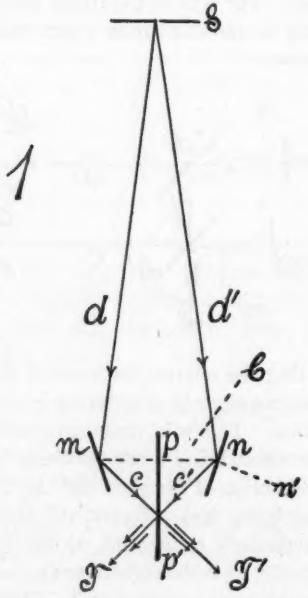
1. The sharp prism heretofore[2] used for cleaving the rays issuing from a collimator (or the slit simply) was dispensed with and the endeavor made to obtain two rays capable of interference. The assemblage of apparatus is shown in figure 1, where S is the slit (to be replaced by a Nernst or a tungsten filament), m and n the opaque mirrors, pp' the half silvered plate. The



rays dd', from S, pass after reflection into c and c' and may be observed by spectrotelescopes placed either at T or T'. In the first experiments the distance dp' was about 4 meters and the distance mn, 10 cm. The mirrors n and pp' were on micrometers with the screws normal to their respective faces. The distance mn must be within the limits of the wedge of light from the slit and is therefore small, unless d is very large. Both pp' and n are on the

rotating rail, whereas m is fixed. The apparatus was also adjustable for reversed rays by attaching an auxiliary mirror, normal to the rays d' prolonged through n. As S is distant, this slit must be long as otherwise the spectrum band will be a mere horizontal line and the fringes difficult to detect. A doublet of lenses each about 10 cm. in diameter, of the same focal power (1/60 cm.) but respectively convex and concave and having a combined focal distance of about 5 or 6 meters, is of advantage for focussing a large solar image (1 to 2 inches in diameter) on the slit. The Nernst or tungsten filament gives the same advantages at once, but the former is too thick.

The fringes are exceedingly difficult to find in spite of the brilliant spectra. It was not until after about three days of searching, in which besides sunlight I used the filaments as well as the methods of direct and of reversed rays that I ultimately succeeded with the former. The fringes lie quite sharply in a definite focal plane, usually between that of the slit image and the principal focal plane. After being found they are strong elliptic spectrum fringes; but when lost nevertheless difficult to rediscover.

The achromatics which coincide in adjustment with horizontal spectrum fringes and are seen with the slit image out of focus, are additionally difficult to find because of the short height of the slit image. As first obtained they lacked brilliancy and were not easily observed.

An attempt was made to register slight lateral displacements of the slit in terms of the displacement of fringes; but as the slit images are thrown out of coincidence when the slit moves, trustworthy numerical data could not be obtained. Incidentally it might appear that two vertical lines of the slit 0.0014 cm. apart should wipe out each other's fringes as a case of interference; but this is not the fact as slit widths over 100 times broader are admissible.

2. After completing these experiments, the distance between slit S and the mirrors mn was increased to about nine meters. The same lens doublet focussing a large solar image on the slit was used as before. When the fringes were found, in view of the longer distance, d, the slit could be opened to over a millimeter of breadth before they quite vanished, from the spectrum.

Operating with two successive slits at about 9 meters from the interferometer, one of which received the light through the other, I found that two independent sets of fringes very different in size and inclination could be put in the field together. Further investigation showed that the size and inclination of the fringes is essentially dependent on the degree of parallelism of the two corresponding slit images. When the images are parallel, the fringes are of maximum size and vertical. When the images are not quite parallel (they incline in opposite directions when the slit is slightly rotated in its own plane from the vertical), the fringes rapidly grow smaller, rotate and vanish. With parallel slit images the spectrum ellipses are centered in the field; otherwise they are very far out of center. The adjustment for actual (not x-like) coincidence must therefore be made with precision, if the fringes are to appear.

Further work was also done with sunlight to obtain more pronounced achromatics. For this purpose a compensator, C, was inserted to equalize the glass path in the half silvered plate. Huge spectrum ellipses were obtained in this way and their centers were placed above the telescopic field so that the fringes seen were large horizontal bars. On removing the spectroscope and placing the slit images out of focus, brilliant achromatics were now obtained, of the concentric hyperbolic type, vividly colored and broad between the apices, and diminishing to hair lines laterally. With these it was possible to enlarge the slit to at least 3 mm., without destroying the fringes, though they became more vague. It is again necessary that the slit images, when in focus, should be quite parallel. It was possible to place a plate of ground glass on the far side of the slit, without destroying the fringes; but not on the side towards the interferometer. In other respects the behavior was as described in the case of achromatics in the earlier experiments with a cleavage prism.

Finally the spectrum fringes and the corresponding achromatics were obtained with the light of a Nernst filament, by focussing an image of it with a strong condenser lens on the slit. The experiments however are very difficult. The spectrum fringes are often weak, out of focus and extremely sensitive to small disadjustments in the horizontal and vertical coincidence of the slit images. They require a fine slit. When well produced the achromatics are also obtainable on removing the spectroscope when the spectrum fringes are horizontal bars. The achromatics may also be obtained brilliantly without the condenser lens; but the adjustment must in such a case be made first with sunlight, as the spectrum from the Nernst filament is too feeble for detecting fringes so elusive as the present. The achromatics however are strong and brilliant even here (Nernst filament).

An interesting result is obtained in case of the achromatic fringes by narrowing one of the beams, for instance that coming from the mirror m (figure 1), by a screen with a slit about 2 mm. wide. In such a case the slit image, out of focus, is correspondingly narrowed. It may be passed from side to side of the broad washed slit image coming from the mirror n, by moving its adjustment screws (vertical axis). The fringes then appear only in a particular position of the narrow image in the field of the broader; but when they do appear they spread far beyond the margins of the narrow image on both sides. Interference thus apparently occurs where but one beam is present. The phenomenon is like those instanced before and means, as I understand it, that the beams have met in some other focal plane, though one is tempted to conclude that interference is stimulated by resonance, in particular as it is often impossible to find a plane in which they have met. The achromatics may sometimes be seen before and behind the principal focal plane, but more frequently either in the one or in the other region, only.

[1] Advance note from a Report to the Carnegie Institution of Washington.

[2] These PROCEEDINGS, 3, 1917 (565).

A STUDY OF THE MOTIONS OF FORTY-EIGHT DOUBLE STARS

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The double stars examined in the following investigation will all be recognized as old and well-known pairs, on each of which a large number of measures have accumulated. Each of these has shown a considerable motion since discovery, but although orbits have been computed for several of them, there is no case in which certain knowledge of the true form of the orbit can be had at this time, nor with the greater number of these pairs is it even certain that the motion is orbital at all. It was, in fact, in view of this uncertainty that the following computations were undertaken. The complete list of double stars at present known to us was examined, and all of those in which, notwithstanding the large number of measures made upon them, it seemed uncertain whether the motion is rectilinear or orbital were selected for study. The primary object was to ascertain, when possible, in which pairs rapid motion during the next few years might be reasonably expected and those in which the distance is now near a maximum value. It is upon such pairs that observations are most urgently needed, while upon those for which the analysis gives no indication that the companion is now near a critical part of the orbit the multiplication of observations is at the present time unnecessary.

It is well known that with the type of pairs here considered, in which less than two quadrants have been described by the companion, a series of widely different orbits may be obtained, any one of which may satisfy the observations within the limits of error naturally to be expected; and this, even although it may afterward appear that the motion is only rectilinear and the pair is not a true binary system. It was therefore thought best not to begin each computation by so adjusting the six constants of the orbit that the observations should be represented by them, but rather to assume that the motion is rectilinear, and having found the straight line which best represents all the observations to examine in each case whether there is definite evidence that the motion of the companion in the straight line is, or is not a uniform motion. If a continuous progressive change is found, greater than can be assumed due to errors of observation, it is evident that the motion is not rectilinear, while in the many cases in which the variations are less, or but little greater, than might be attributed to errors of observation the evidence for the binary character of the system must be regarded as inconclusive.

The first step in each case was to form suitable means of the numerous observations. Among the earlier measures, those of Maedler and also those of the few occasional observers were rejected, especially since in no case were there wanting accurate observations by Σ, 0Σ and Δ during the same intervals. Practically all modern observations were, however, included, and these were

weighted in proportion to the number of nights, except that at least double weight was always given to the measures of Aitken and Hussey with the 36-inch and to those of Burnham with the 40-inch.

The approximate rectilinear path having been found, the differential corrections Δv, ΔT and Δd to the mean motion, the time of closest approach and the direction, respectively, were next found by a least square adjustment, measures of angle, only, being employed; the distance at closest approach was next found from the weighted means of the measured angles and distances. It was assumed that the line as thus finally adjusted was the best determination of the path obtainable upon the hypothesis that the path is rectilinear. The average velocity, v, along the path was finally compared with the observed velocity at different epochs in order to ascertain any systematic variation that might exist.

It is found convenient to group the pairs examined into five classes, determined by the results of the investigation as follows:

Class A.—The velocity shows a decided and certain increase. Rapid motion in angle is to be expected, and careful measures, particularly of angle, are especially needed at this time.

0Σ 20, (Burnham's General Catalogue, No. **479**). Though for 50 years after discovery the velocity remained nearly constant, (about 0″.016), it has, since 1897, very decidedly increased, (to 0″.036), and orbital motion is certain. The period will, however, much exceed the value, 136.2 years, assigned by Glasenapp.

Σ 208, (**1074**). It was found quite impossible to represent both the earlier and the later observations by a single rectilinear path. The velocity is now nearly three times the average velocity and the stars are near their minimum distance.

Σ 963, (**3625**). The motion is here very slow, but there is a systematic increase in v, from 0″.0070 in 1839 to 0″.0131 in 1908.

Σ 1306, (**4923**). The observations from 1845–'65 are discordant but there is a remarkable increase in v during recent years. Undoubtedly binary, and now so near the least distance that an entry into the first quadrant may be expected during the next few years. The brighter component (5.0 magnitude) is one of Newcomb's Fundamental Stars and is contained in the American Ephemeris. As we, of course, know nothing of the location of the center of gravity between this and its 8.5 magnitude companion, it would seem that this is unsuitable for use as a fundamental star.

Σ 1834, (**6832**). The distance in this pair is now but little greater than 0″.1 and the measures since 1892 are hopelessly inconsistent. Analysis of the motion from discovery until the date mentioned shows a steady, and finally rapid, increase in velocity. The angle of the companion has probably increased 180° during the past few years.

Σ 113, (**707**) and Σ 2576, (**9602**), also show a definite and certain increase of velocity in recent years.

Class B.—The velocity shows a definite diminution. The stars are nearly at their maximum distance. Measures are now needed to fix the position of apastron, but the angle will probably change but little. The distance should be measured with special care.

Σ 305, (1427); Σ 1457, (5508); 0Σ 261, (6415); and 0Σ 288, (7049) all show a decided and progressive diminution of the velocity. In the last, (0Σ 288), it is, in fact, uncertain whether the motion did not wholly cease about 1912, and the two stars begin to draw together.

Class C.—A systematic and certain variation of the velocity, but the companion is not certainly at either periastron or at apastron.

The most striking examples in this class are: Σ 400, (1747), in which the velocity has been steadily increasing since discovery; 0Σ 215, (5365), in which the analysis shows the maximum velocity to have been attained about 1893; Σ 1865, (6955), in which the velocity mounted quite suddenly to a value nearly four times the average about 1897, when the stars were closest together, and which has since been rapidly diminishing, and Σ 2118, (7834), in which the maximum velocity was attained about 1895. In all of these cases the rise and fall of the velocity is very striking. There were also found to belong to this class,—Σ 2, (21); Σ 535, (2161); Σ 1536, (5765); and Σ 1687, (6296).

Class D.—The velocity apparently varies, (sometimes quite irregularly), and orbital motion is strongly suggested, but it is in no case entirely certain. But few observations are required on these pairs at this time.

Fourteen of the 48 pairs were found to belong to this class in which, though orbital motion might be strongly suspected, it is, nevertheless, not yet entirely certain. These pairs are,—

Σ 13, (92),	0Σ 159, (3678),	Σ 1757, (6530),
Σ 138, (830),	Σ 1175, (4402),	Σ 2315, (8548),
0Σ 50, (1568),	Σ 1338, (5030),	Σ 2434, (8986),
Σ 460, (1952),	Σ 1429, (5421),	0Σ 413, (10533),
	Σ 1517, (5707),	0Σ 437, (10922).

Class E.—No certain trace of orbital motion. A multiplication of observations on these pairs is unnecessary.

Fifteen pairs showed no certain trace of orbital motion:

H 1968, (216),	0Σ 92, (2445),	Σ 1643, (6174),
0Σ 18, (374),	*H* 3823, (3112),	Σ 1883, (7013),
0Σ 43, (1365),	Σ 1104, (4098),	0Σ 297, (7320),
Σ 367, (1623),	Σ 1423, (5385),	Σ 2199, (8118),
Σ 567, (2272),	0Σ 237, (5859),	Σ 2574, (9570).

Summary.—The final summary for the 48 pairs examined is as follows:

Class A....7 Pairs, Class B....4 Pairs, Class C....8 Pairs.

Total pairs in which orbital motion is certain.....19

Class D....14 Pairs, Class E....15 Pairs

Total pairs in which the evidence thus far is negative.....29

THE STRUCTURE OF AN ELECTROMAGNETIC FIELD

By H. Bateman

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Communicated by A. A. Noyes, March 18, 1918

1. Starting with the fundamental equations

$$c \operatorname{rot} \mathbf{H} = \frac{\partial \mathbf{E}}{\partial t} + \rho \mathbf{v}, \qquad \operatorname{div} \mathbf{E} = \rho,$$

$$c \operatorname{rot} \mathbf{E} = -\frac{\partial \mathbf{H}}{\partial t}, \qquad \operatorname{div} \mathbf{H} = 0,$$

we adopt as an elementary solution

$$\mathbf{H} = \operatorname{rot} \mathbf{A}, \ \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla \Phi, \ \Psi = \operatorname{div} \mathbf{A} + \frac{1}{c}\frac{\partial \Phi}{\partial t}, \ \rho = -\frac{1}{c}\frac{\partial \Psi}{\partial t}, \ \rho \mathbf{v} = c \nabla \Psi,$$

where

$$\mathbf{A} = c \int_{-\infty}^{\alpha} f(\tau) \nabla \log \left[\mathbf{s}.\mathbf{r} - c(t - \tau)\right] d\tau,$$

$$\Phi = -\int_{-\infty}^{\alpha} f(\tau) \frac{\partial}{\partial t} \log \left[\mathbf{s}.\mathbf{r} - c(t - \tau)\right] d\tau.$$

In these expressions $\mathbf{s}$ is a unit vector depending on τ, $\mathbf{r}$ denotes the radius vector from the point with co-ordinates $\xi(\tau)$, $\eta(\tau)$ $\zeta(\tau)$, to the point with co-ordinates x, y, z, while α is defined by the equation

$$[x - \xi(\alpha)]^2 + [y - \eta(\alpha)]^2 + [x - \zeta(\alpha)]^2 = c^2 (t - \alpha)^2; \ \alpha \leqq t.$$

The function Ψ is given by the formula

$$\Psi = \frac{c}{\nu} f(\alpha),$$

where

$$\nu = \xi'(\alpha)[x - \xi(\alpha)] + \eta'(\alpha)[y - \eta(\alpha)] + \zeta'(\alpha)[z - \zeta(\alpha)] - c^2 (t - \alpha).$$

This elementary field corresponds to a state of affairs in which electric charges of a concentrated form are created and travel along straight lines with the velocity of light, the directions of these lines being specified by the different values of the unit vector $\mathbf{s}$. Whenever a concentrated electric charge is created an amount of electricity which will just compensate it is fired out in all directions and provides an elementary 'aether' which is the seat of the electromagnetic field of the concentrated charge. A concentrated electric charge and its elementary aether lie at any instant on a sphere whose centre is at the point where these charges originated[1]; if now this point moves with a velocity

less than the velocity of light the different spheres bearing electricity that exist at time t do not intersect and if the arbitrary function $f(\alpha)$ is never zero there will be a sphere through each point of space so that our elementary aethers will fill the whole of space; if however the function $f(\alpha)$ is sometimes zero, for axemple if it is zero when α is less than α_0, then the elementary aethers will not fill the whole of space.

If we subtract from the above field another one of the same type in which the unit vector function $\mathbf{s}$ has a different value we obtain a field in which $\rho\mathbf{v}$ and ρ are zero except in the neighbourhood of the concentrated electric charges, there is thus a cancelling of electricity when the two elementary aethers are superposed and we get an aether in the ordinary sense of the word. The field is now one in which concentrated charges of opposite signs are continually produced by a process of separation analogous to that described by Heaviside in 1901. The field thus obtained belongs to the type in which there is a rectilinear flow of energy and no accumulation of energy at any point of space: the energy in such a field may therefore be regarded as kinetic energy or energy of motion.

The most general field that possesses the property just mentioned and the additional property that the volume charge and current in the aether are zero outside the singularities of the field is obtained by writing

$$\mathbf{M} = \mathbf{H} + i\mathbf{E} = cF(\alpha,\beta)\{\nabla\alpha \times \nabla\beta\} = iF(\alpha,\beta)\left\{\frac{\partial\beta}{\partial t}\nabla\alpha - \frac{\partial\alpha}{\partial t}\nabla\beta\right\},$$

where α and β are defined by equations of type

$$z - ct = f(\alpha,\beta) + (x + iy)\,\theta(\alpha,\beta), \quad z + ct = g(\alpha,\beta) - \frac{x - iy}{\theta(\alpha,\beta)},$$

f, g and θ being arbitrary functions of α and β. It may be remarked that $\mathbf{M}.\mathbf{M} = 0$ and that θ is a solution of the wave equation.

In all the fields of the above type electricity or magnetism travels along straight lines with the velocity of light (the case of a plane wave of light is, however, an exception). To obtain fields in which electricity or magnetism travels with a velocity less than that of light we must superpose fields of the above type in such a way that there is a cancelling of nearly all the concentrated electric or magnetic charges. It is fairly easy to prove that the field of an isolated electric pole moving with a velocity less than that of light can be regarded as the limit of two superposed radiant fields of the type obtained by subtracting two of our elementary solutions. According to this idea the electricity at an electric pole is continually being renewed, moreover, it is the electric charge itself which is directly responsible for the effects produced at a distance, but to understand fully the production of these effects we must consider how this charge is constituted remembering how the field

was built up from four of our elementary fields. It is important to notice in this connection that the volume density of electricity and electric current in an elementary field are independent of the direction in which the concentrated charges move; this simplifies matters when we want to ascertain the structure of the aether for an electric pole that is built up in a specified way.

When a number of radiant fields are superposed a cancelling of concentrated electric charges seems to be necessary in order that the principle of the conservation of energy may hold and in order that steady states of motion may be possible. Even if the cancelling is not complete everywhere it must at least be sufficient to prevent any free electricity from going to infinity. If this less stringent condition is adopted it is possible to admit fields in which the charge associated with an electric pole fluctuates slightly and electric charges are fired from one electric pole to another. It may be possible to explain gravitation in this way for it will soon be realised that the necessary fluctuation of charge is exceedingly small. The question may also be raised whether it is necessary for there to be a complete cancelling of the charges in the elementary aethers when a number of elementary fields are superposed. If the answer is in the negative the volume densities of electricity and convection current will be derivable from a function Ψ which satisfies the wave-equation at points not occupied by matter.

Summing up the essential features of the present theory, we may say that all electric charges are supposed to really travel along rectilinear paths with the velocity of light; this implies that when electricity appears to move with a smaller velocity it is made up of different entities at different times being constantly renewed so to speak. The fact that an electric charge which has been moving along a rectilinear path with the velocity of light has no surrounding field[2] is quite consistent with the present view, for all electric charge arises from electric separation[3] and its aether is created in the process, consequently an electric charge which has moved along a rectilinear path for all time with the velocity of light would have no aether to support a field.

When we admit the mathematical possibility that the electric charges in the universe have not existed in the free state for the whole of time we find that it is by no means certain that the aether fills the whole of space and this raises some interesting philosophical questions. Some of the logical difficulties[4] in the ideas of contact action and action at a distance are avoided in the present theory because an electric charge at a point P produces an effect at a distance point Q for the simple reason that either a portion of the charge itself, or a portion of the compensating charge that was created at the same time, actually goes to Q and helps to produce the particle that is acted upon, or rather the entity that represents the particle at the moment under consideration.

Passing on to a brief consideration of some more familiar types of fields we shall superpose the fields of point charges using Lienard's potentials

$$\mathbf{A} = -\frac{e}{4\pi}\frac{\mathbf{v}(\alpha)}{\nu}, \qquad \Phi = -\frac{e}{4\pi}\frac{c}{\nu}.$$

We shall be interested chiefly in the effect of an operation analogous to differentiation. Let us suppose that the co-ordinates of the point charge at time α depend on a parameter β as well as α and let $\mathbf{b}$ denote the vector with components $\partial\xi/\partial\beta$, $\partial\eta/\partial\beta$, $\partial\zeta/\partial\beta$, then if $\mathbf{f}$ is a function of α and β it is easy to prove that

$$\frac{\partial}{\partial\beta}\left(\frac{f}{\nu}\right) = \frac{1}{\nu}\frac{\partial f}{\partial\beta} - \operatorname{div}\left(\frac{f\mathbf{b}}{\nu}\right).$$

When $f = \partial\omega\,(\alpha,\beta)/\partial\alpha$ this formula may be written in a more convenient form by making use of the relations

$$\frac{\partial}{\partial x}\left(\frac{\xi'}{\nu}\right) + \frac{\partial}{\partial y}\left(\frac{\eta'}{\nu}\right) + \frac{\partial}{\partial z}\left(\frac{\zeta'}{\nu}\right) + \frac{\partial}{\partial t}\left(\frac{1}{\nu}\right) = 0,$$

$$\xi'\frac{\partial}{\partial x} + \eta'\frac{\partial}{\partial y} + \zeta'\frac{\partial}{\partial z} + \frac{\partial}{\partial t} = \frac{\partial}{\partial\alpha},$$

where the operators in the last equation are supposed to act on a function of α and β.

We thus obtain the equation

$$\frac{\partial}{\partial\beta}\left(\frac{1}{\nu}\frac{\partial\omega}{\partial\alpha}\right) = \frac{\partial}{\partial x}\left[\frac{1}{\nu}\frac{\partial(\omega,\xi)}{\partial(\beta,\alpha)}\right] + \frac{\partial}{\partial y}\left[\frac{1}{\nu}\frac{\partial(\omega,\eta)}{\partial(\beta,\alpha)}\right] + \frac{\partial}{\partial z}\left[\frac{1}{\nu}\frac{\partial(\omega,\zeta)}{\partial(\beta,\alpha)}\right] + \frac{\partial}{\partial t}\left[\frac{1}{\nu}\frac{\partial(\omega,\alpha)}{\partial(\beta,\alpha)}\right].$$

In particular we have

$$\frac{\partial A_x}{\partial\beta} = -\frac{e}{4\pi}\left[\frac{\partial}{\partial y}\left\{\frac{1}{\nu}\left(\eta'\frac{\partial\xi}{\partial\beta} - \xi'\frac{\partial\eta}{\partial\beta}\right)\right\} - \frac{\partial}{\partial z}\left\{\frac{1}{\nu}\left(\xi'\frac{\partial\zeta}{\partial\beta} - \xi'\frac{\partial\xi}{\partial\beta}\right)\right\} + \frac{\partial}{\partial t}\left(\frac{1}{\nu}\frac{\partial\xi}{\partial\beta}\right)\right],$$

$$\frac{\partial\Phi}{\partial\beta} = \frac{ec}{4\pi}\left[\frac{\partial}{\partial x}\left\{\frac{1}{\nu}\frac{\partial\xi}{\partial\beta}\right\} + \frac{\partial}{\partial y}\left\{\frac{1}{\nu}\frac{\partial\eta}{\partial\beta}\right\} + \frac{\partial}{\partial z}\left\{\frac{1}{\nu}\frac{\partial\zeta}{\partial\beta}\right\}\right],$$

Writing $\mathbf{p} = e\mathbf{b}/4\pi$, we find that for an electric doublet of moment $4\pi\mathbf{p}$ the electric potentials are

$$\mathbf{A} = \operatorname{rot}\left\{\frac{1}{\nu}(\mathbf{v}\times\mathbf{p})\right\} - \frac{\partial}{\partial t}\left\{\frac{1}{\nu}\mathbf{p}\right\}, \quad \Phi = c\operatorname{div}\left\{\frac{1}{\nu}\mathbf{p}\right\}.$$

This of course is a simple generalisation of the well known result due to Hertz and Righi.

It is well known and easy to verify that the same electromagnetic field may be derived from the magnetic potentials.

$$\mathbf{B} = -c\operatorname{rot}\left\{\frac{1}{\nu}\mathbf{p}\right\} - \frac{1}{c}\frac{\partial}{\partial t}\left\{\frac{1}{\nu}(\mathbf{v}\times\mathbf{p})\right\}, \quad \Omega = \operatorname{div}\left\{\frac{1}{\nu}(\mathbf{v}\times\mathbf{p})\right\},$$

with the aid of the formulae

$$\mathbf{E} = \text{rot}\,\mathbf{B}, \qquad \mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} + \nabla\Omega.$$

Let us now write

$$\mathbf{M} = \mathbf{H} + i\mathbf{E} = i\,\text{rot}\,\mathbf{L} \equiv \frac{1}{c}\frac{\partial \mathbf{L}}{\partial t} + \nabla\Lambda,$$

$$\mathbf{L} = \mathbf{B} - i\mathbf{A} = \frac{\partial \mathbf{G}}{\partial t} + ic\,\text{rot}\,\mathbf{G}, \qquad \Lambda = \Omega - i\Phi = -c\,\text{div}\,\mathbf{G};$$

then in the case of an electric doublet we have

$$\mathbf{G} = \frac{i}{\nu}\left[\mathbf{p} + \frac{i}{c}(\mathbf{v}\times\mathbf{p})\right].$$

To obtain the field of a magnetic doublet we write $i\mathbf{q}$ instead of $\mathbf{p}$ for the field of a magnetic pole of strength μ is derived from the magnetic potentials

$$\mathbf{B} = \frac{\mu}{4\pi}\frac{\mathbf{v}}{\nu}, \qquad \Omega = \frac{\mu}{4\pi}\frac{c}{\nu}.$$

If $\mathbf{m} = \mathbf{q} - i\mathbf{p}$ and

$$\mathbf{G} = -\frac{1}{\nu}\left[\mathbf{m} + \frac{i}{c}(\mathbf{v}\times\mathbf{m})\right] = \frac{\mathbf{g}(\alpha)}{\nu}, \text{ say,}$$

the derived field is that of an electric doublet and magnetic doublet which move together. When the vector $\mathbf{g}$ is given it is easy to determine the moment $4\pi\mathbf{p}$ of the electric doublet and the moment $4\pi\mathbf{q}$ of the magnetic doublet.

The function $\mathbf{g}(\alpha)/\nu$ may be regarded as analogous to the fundamental potential function $1/r$ of electrostatics and Hertzian functions of higher order may be derived from it by differentiation just as potential functions involving spherical harmonics are derived from $1/r$ by differentiation according to a method developed by Maxwell.

Let us regard $\xi, \eta, \zeta, \mathbf{g}$ as functions of α and a parameter β then we obtain by differentiation a new Hertzian function whose x-component is

$$G_x' = \frac{\partial}{\partial\beta}\left\{\frac{1}{\nu}g_x(\alpha)\right\} = \frac{1}{\nu}\frac{\partial g_x}{\partial\beta} - \text{div}\left\{\frac{\mathbf{b}g_x}{\nu}\right\}.$$

It should be noticed that this expression for G_x' contains differentiation with regard to x, y and z but not t so that there is apparently a lack of symmetry. This is due to the fact that α is taken to be independent of β; we can easily introduce a term involving a differentiation with regard to t by making use of the identity

$$\frac{1}{\nu}f'(\alpha) = \text{div}\left(\frac{f\mathbf{v}}{\nu}\right) + \frac{\partial}{\partial t}\left(\frac{f}{\nu}\right),$$

but it is generally simpler and more convenient to retain the former expression.

The process of differentiation may be carried out any number of times with respect to different parameters using formulae of differentiation analogous to the above. When the various derivatives are added together the result indicates that the natural generalisation of a series of spherical harmonics of form

$$\frac{S_0(\theta,\phi)}{r} + \frac{S_1(\theta,\phi)}{r^2} + \frac{S_2(\theta,\phi)}{r^3} +$$

is the following type of series of Hertzian functions of different orders

$$\mathbf{G} = \frac{\mathbf{g}_0}{\nu} + \operatorname{div}\left(\frac{\mathbf{a}_0\mathbf{a}_1}{\nu}\right) + \operatorname{div}\operatorname{div}\left(\frac{\mathbf{b}_0\mathbf{b}_1\mathbf{b}_2}{\nu}\right) + \operatorname{div}\operatorname{div}\operatorname{div}\left(\frac{\mathbf{c}_0\mathbf{c}_1\mathbf{c}_2\mathbf{c}_3}{\nu}\right) + .$$

Here $\mathbf{g}_0$, $\mathbf{a}_0$, $\mathbf{a}_1$, $\mathbf{b}_0$, $\mathbf{b}_1$, $\mathbf{b}_2$ .... are arbitrary vector functions of α. It should be remarked that the vector with suffix n is treated as the vector in forming the n divergence while the other vectors are treated for the moment as scalar quantities. The product of k vectors which occurs in the $(k+1)$th term is to be regarded as a tensor of the kth order with k components each of which is a product of components of the separate vectors; there appear to be enough arbitrary functions in a sum of products of this type with $k = 0, 1, 2, \ldots, K$ for the representation of the sum of a number of Hertzian functions up to order K.

[1] As each shell of electricity moves outwards it induces a secondary separation of electricity so that electricity flows back to a new position of the primary singularity (ξ, η, ζ) and tends to maintain the electric separation. The volume density of the compensating electricity created at the primary singularity isthus not ρ but is proportional to Ψ/r, it is this electricity which is regarded as forming the elementary æther associated with the primary singularity and it is this electricity which, on account of its displacement from the concentrated charge, is directly responsible for the field.

[2] See for instance Wilson, E. B., *Washington Acad. Sci.*, 6, 1916, (665–669).

[3] Larmor, J., *London, Proc. Mathe. Soc.*, 13, 1913, p. 51.

[4] Whitehead, A. N., *The Anatomy of some Scientific Ideas, The Organization of Thought*, London, 1917, p. 182.

INVARIANTS WHICH ARE FUNCTIONS OF PARAMETERS OF THE TRANSFORMATION

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A systematic theory and interpretation of invariantive functions which contain the parameters of the linear transformations to which a quantic

f_m of order m is subjected has not been formulated, although a paper on invariants[1] published in 1843 by Boole treated certain functions of this type. These were the concomitants of forms under transformations which rotate cartesian axes inclined at an angle w into another set with inclination w'; invariants which contain the parameter w. Orthogonal concomitants, which are special cases with $w = \frac{1}{2}\pi$, were made the subject of a number of later papers[2], notably by Elliott and by MacMahon.

I have considered a general doctrine of such concomitants for the transformation with four parameters

$$T: x_1 = \alpha_1 x_1' + \alpha_2 x_2', x_2 = \beta_0 x_1' + \beta_1 x_2'; \quad D = \alpha_1\beta_1 - \alpha_2\beta_0 \neq 0.$$

The elements of the methods are based upon the two forms

$$\xi = 2\beta_0 x_1 + (\beta_1 - \alpha_1 + \Delta)x_2, \quad \eta = 2\beta_0 x_1 - (\beta_1 - \alpha_1 - \Delta)x_2;$$

whose roots are the poles of T, and the expansion of f_m in terms of ξ, η as arguments. The quantity Δ employed here is the square root of the discriminant of the form

$$J: \beta_0 x_1^2 + (\beta_1 - \alpha_1)x_1x_2 - \alpha_2 x_2^2.$$

The coefficients φ_{m-2i} $(i = 0, \ldots, m)$ in the expansion of f_m are invariants of a new type[3] belonging to the domain $R(1, T, \Delta)$ of rational polynomials in the coefficients of f_m and those of T, increased by adjunction of Δ. These invariants compose a fundamental system in R. They satisfy the invariant relations

$$\varphi'_{m-2i} = \rho^{m-2i} D^i \varphi_{m-2i} \; (i = 0, \ldots, m), \tag{1}$$

in which ρ is one of the two factors of D in R:

$$\rho = \tfrac{1}{2}(\alpha_1 + \beta_1 + \Delta). \tag{2}$$

When one seeks complete systems for the given domain $R(1, T, 0)$, free from Δ but including rationally the coefficients of T, it is found that the concomitants are in one to one correspondence with those invariantive products

$$P = \prod_{i=0}^{m} \varphi_{m-2i}^{x_i} \xi^{\alpha} \eta^{\beta}$$

for which the exponent of ρ in the invariant relation $P' = \rho^a D^b P$, is zero. The conclusion is then drawn that concomitants in $R(1, T, 0)$ are in one to one correspondence with the solutions of the diophantine equation

$$\delta \equiv \sum_{i=0}^{m} x_i (m - 2i) + \beta - \alpha = 0. \tag{3}$$

This infinitude of concomitants therefore forms a system which possesses the property of finiteness,[4] and the fundamental invariants are furnished by the finite set of irreducible solutions of $\delta = 0$.

In the ternary realm the lines joining the three poles of the transformation T on three variables furnish three linear forms in terms of which any quantic f_m in three variables can be expanded. The coefficients in this expansion urnish complete systems in each of several domains. In particular, if T is the transformation which rotates cartesian axes in three dimensional space,

$$\begin{aligned} x &= l_1x' + l_2y' + l_3z', \\ y &= m_1x' + m_2y' + m_3z', \\ z &= n_1x' + n_2y' + n_3z', \end{aligned}$$

the coefficients being the well-known direction cosines of three axes, the invariant triangle on the poles consists of the lines

$$\begin{aligned} f_{\pm 1} &\equiv (l_3 + n_1 e^{\mp i\theta})\, x + (m_3 + n_2 e^{\mp i\theta})\, y + (n_3 - \overline{l_1 + m_2} e^{\mp i\theta} + e^{\mp 2i\theta})\, z = 0, \\ f_0 &\equiv (l_3 + n_1)\, x + (m_3 + n_2)\, y + (n_3 - l_1 - m_2 + 1)\, z = 0, \end{aligned}$$

where θ is a definite auxiliary angle. The coefficients in the expansion of f_m in the arguments $f_{\pm 1}$, f_0 are invariants belonging to the domain of complex numbers, while the finiteness of complete systems in the real domain is determined, and the fundamental concomitants are given, by the finite set of irreducible solutions of the linear diophantine equation

$$\sum_{i=0}^{m} \sum_{j=0}^{m-i} x_j^{(i)} (m - i - 2j) + \beta - \alpha = 0. \tag{4}$$

The Invariants of Relativity.—Among numerous important particular cases of the above theory is the transformation of space and time coordinates in the theory of relativity, known as the transformation of Einstein.[5] This consists of

$$T_1 : \; t = \mu (c^2t' + vx')/c, \;\; x = \mu (vt' + x')c, \;\; y = y', \;\; z = z',$$

where $\mu = 1/(c^2 - v^2)^{-\frac{1}{2}}$, t is the time, c the velocity of light, v the relative velocity of the moving systems of reference and x, y, z space coördinates. For these unitary substitutions, $\xi = ct + x$, $\eta = ct - x$ and the invariant relations are

$$\xi' = \rho\xi, \; \eta' = \rho^{-1}\eta,$$

in which (Cf. (2))

$$\rho = \sqrt{c - v} / \sqrt{c + v}.$$

We now find

$$J : c^2t^2 - x^2,$$

this being an absolute universal covariant of T_1 for all values of the relative velocity V.

A binary form $f_{1m}(t, x)$ in t and x, whose coefficients are constants or arbitrary functions of the quantities left fixed by T_1, has a finite system of non-absolute invariants corresponding to the forms in the system for f_m belonging to R $(1, T, \Delta)$, and a finite system of absolute concomitants analogous to the system for f_m in the domain R $(1, T, 0)$. To obtain the concomitants of $f_{1m}(t, x)$ one may either particularize those of f_m under T, making the substitutions which reduce T to T_1, or, the invariants under Einstein's transformations can be developed *ab initio* by the methods described above for T, the arguments of the expansion of f_{1m} being now $\xi = ct + x$, $\eta = ct - x$. These invariantive functions represent invariant loci in four dimensional space if the time t is interpreted as a fourth dimension. All are free from v.

A paper in which the above theory and applications are developed in detail and which includes tables of the relativity invariants computed for the general f_{1m} in the case of the non-absolute systems, and for the orders 1 to 3, inclusive, in the case of the absolute systems, is to appear in the *Annals of Mathematics*.

[1] Boole, *Cambridge Mathematical Journal*, 3, 1843, (1).

[2] Elliott, *London, Proc. Math. Soc.*, 33, 1901, (226).

[3] O. E. Glenn, *New York. Trans. Amer. Math. Soc.*, 18, 1917. (443)

[4] Hilbert, *Leipzig, Math. Ann.*, 36, 1890, (473).

[5] Einstein, *Leipzig, Ann. Physik*, 17, 1905. Lorentz, Einstein, and Minkowski, *Das Relativitätsprinzip*, 1913, p. 27. R. D. Carmichael, *The Theory of Relativity*, 1913, p. 44.